A DISCRETE COLLOCATION METHOD FOR SYMM'S INTEGRAL EQUATION ON CURVES WITH CORNERS

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Abstract

This paper is devoted to the approximate solution of the classical first-kind boundary integral equation with logarithmic kernel (Symm's equation) on a closed polygonal boundary in $\mathbb{R}^2$. We propose a fully discrete method with a trial space of trigonometric polynomials, combined with a trapezoidal rule approximation of the integrals. Before discretization the equation is transformed using a nonlinear (mesh grading) parametrization of the boundary curve which has the effect of smoothing out the singularities at the corners and yields fast convergence of the approximate solutions. The convergence results are illustrated with some numerical examples.

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1 Introduction

This work is concerned with the numerical solution of

$$-\frac{1}{\pi}\int_{\Gamma} \log|x-\xi|u(\xi)\, d\Gamma(\xi) = f(x), \qquad x \in \Gamma, \tag{1.1}$$

with Γ the boundary of a simply connected bounded domain Ω in $\mathbb{R}^2$. Equation (1.1) arises in solving the Dirichlet problem for Laplace's equation on Ω, using boundary integral equation methods.

For the case of smooth Γ, there is now a large literature on the approximate solution of (1.1) by collocation and quadrature methods based on splines or trigonometric polynomials; see [11] for a review. Until recently only special results for low order methods were known when Γ is a polygonal boundary; see [4] for a discussion. With the recent paper [4], results on stability and optimal convergence for spline collocation methods of arbitrarily high order are now available for polygonal Γ. Analogous results for a fully discrete version of the method in [4] were obtained in [5], and a convergence theory for the qualocation method on a polygon was presented in [6].

The purpose of the present paper is to extend the discrete trigonometric collocation method, considered in [1] for smooth closed curves and in [2] for smooth open arcs, to a rapidly convergent method for curves with corners. In addition, this method is easier to implement than the quadrature-collocation scheme of [5]. A generalization of the discrete qualocation methods introduced in [12] to the case of a polygonal boundary can be found in [8].

In this paper, we consider the case that Γ is (infinitely) smooth, with the exception of a corner at a point x_0. In the analysis we further assume that Γ in the neighbourhood of the corner x_0 consists of two straight lines intersecting with an interior angle $(1-\chi)\pi, 0 < |\chi| < 1$. This is believed to be an inessential restriction. The extension to boundary curves with more than one corner is straightforward, see [4, 5]. We assume throughout that the transfinite diameter of Γ is not equal to 1, so that (1.1) is uniquely solvable.

Following the development in [4, 5], we now rewrite (1.1) using an appropriate nonlinear parametrization $\gamma : [0,1] \to \Gamma$ which varies more slowly than arc-length parametrization in the vincinity of x_0. Consider a parametrization $\gamma_0 : [0,1] \to \Gamma$ such that $\gamma_0(0) = \gamma_0(1) = x_0$ and $|\gamma_0'(s)| > 0$ for all $0 \le s \le 1$. Choosing a grading exponent $q \in \mathbb{N}$ and selecting a function ν such that

$$\nu \in C^\infty[0,1], \quad \nu(0) = 0, \quad \nu(1) = 1, \quad \nu'(s) > 0, \quad 0 \le s \le 1, \tag{1.2}$$

we define the mesh grading transformation

$$\gamma(s) = \gamma_0(\omega(s)), \quad \text{where} \quad \omega(s) = \frac{\nu^q(s)}{\nu^q(s) + \nu^q(1-s)}. \tag{1.3}$$

The parametrization γ we have chosen is graded with exponent q near the corner. The simplest choice of ν satisfying (1.2) is, of course, $\nu(s) = s$. A more practical choice of ν in (1.3) is the cubic polynomial

$$\nu(s) = (\frac{1}{q} - \frac{1}{2})(1-2s)^3 + \frac{1}{q}(2s-1) + \frac{1}{2} \tag{1.4}$$

where the grading exponent is an integer ≥ 2, see [9].

Using the change of variables $x = \gamma(s)$, $\xi = \gamma(\sigma)$, Equation (1.1) becomes

$$Kw(s) := -\frac{1}{\pi}\int_0^1 \log|\gamma(s) - \gamma(\sigma)| w(\sigma)\, d\sigma = g(s), \qquad s \in [0,1], \tag{1.5}$$

where

$$w(\sigma) = (2\pi)^{-1}|\gamma'(\sigma)|u(\gamma(\sigma)), \qquad g(s) = f(\gamma(s)). \tag{1.6}$$

The solution w of the transformed equation (1.5) may be made as smooth as desired on $[0,1]$ provided f is smooth and the grading exponent is sufficiently large, and hence w can be optimally approximated using trigonometric polynomials as basis functions.

We decompose (1.5) as

$$Aw + Bw = g \tag{1.7}$$

with

$$Aw(s) = -2\int_0^1 \log|2e^{-1/2}\sin(\pi(s-\sigma))|w(\sigma)\,d\sigma, \tag{1.8}$$

$$Bw(s) = \int_0^1 b(s,\sigma)w(\sigma)\,d\sigma, \tag{1.9}$$

$$b(s,\sigma) := -2\log\left|\frac{\gamma(s)-\gamma(\sigma)}{2e^{-1/2}\sin(\pi(s-\sigma))}\right|, \qquad 0 < s,\sigma < 1, \quad s \neq \sigma. \tag{1.10}$$

The kernel function (1.10) is 1-periodic in both variables and C^∞ for $0 < s,\sigma < 1$, but in contrast to the case of smooth Γ it has fixed singularities at the four corners of the square $[0,1]\times[0,1]$. The operator A arises from studying (1.1) on a circle with radius $e^{-1/2}$.

Applying the analysis of the transformed equation in [4, 5] and using the fact that the eigenfunctions of A are the trigonometric functions, we consider a collocation method with trigonometric trial functions for solving (1.7) in § 2. In § 3 we introduce and analyze a corresponding discrete collocation method. Numerical examples are given in § 4.

2 Trigonometric Collocation

Let $H^t, t \in \mathbb{R}$, be the usual Sobolev spaces of 1-periodic functions (distributions) on the real line, with norm given by

$$\|v\|_t^2 = |\hat{v}(0)|^2 + \sum_{m\neq 0} |m|^{2t}|\hat{v}(m)|^2,$$

where the Fourier coefficients of v are defined by

$$\hat{v}(m) = (v, e^{i2\pi ms}) = \int_0^1 v(s)e^{-i2\pi ms}\,ds.$$

Introduce the collocation points

$$s_j = jh + h/2, \quad j \in \mathbb{Z}, \quad \text{where} \quad h := 1/(2n+1), \tag{2.1}$$

and let $\mathcal{T}_h$ denote the space of trigonometric polynomials of degree $\leq n$ with the standard basis

$$\varphi_k(s) = e^{i2\pi ks}, \quad |k| \leq n. \tag{2.2}$$

Then, for any continuous 1-periodic function v, the interpolatory projection $Q_h v$ onto $\mathcal{T}_h$ is well defined by

$$(Q_h v)(s_j) = v(s_j), \quad j = 0, ..., 2n, \tag{2.3}$$

and satisfies [2]

$$\|v - Q_h v\|_t \leq ch^{r-t}\|v\|_r, \quad v \in H^r, \text{ for } r > 1/2, \quad r \geq t \geq 0. \tag{2.4}$$

Note that, using the basis (2.2), the projection Q_h is given by

$$Q_h v(s) = \sum_{k=-n}^{n} \alpha_k \varphi_k(s), \qquad \alpha_k := h \sum_{j=-n}^{n} v(s_j)\overline{\varphi_k(s_j)};$$

see [1] or [10, Chap. 2.3].

The collocation method for (1.7) consists of solving

$$Q_h(A+B)w_h = Q_h g, \qquad w_h \in \mathcal{T}_h,$$

and since Q_h commutes with A on $\mathcal{T}_h$, we have the equivalent formulation

$$(A + Q_h B)w_h = Q_h g, \qquad w_h \in \mathcal{T}_h. \tag{2.5}$$

Following [1, 4], we rewrite (1.7) as the second kind equation

$$(I + M)w = e, \qquad \text{with} \quad M = A^{-1}B, \quad e = A^{-1}g. \tag{2.6}$$

Recall [1] that the operator A of (1.8) takes the form

$$Av(s) = \sum_{m \in \mathbb{Z}} \frac{\hat{v}(m)}{\max(1, |m|)} \varphi_m(s)$$

and is an isomorphism of H^t onto H^{t+1} for any real t, and its inverse satisfies

$$A^{-1} = -\mathcal{H}D + \mathcal{J} = -D\mathcal{H} + \mathcal{J} \tag{2.7}$$

with $Dv(s) = v'(s)$, $\mathcal{J}v(s) = \hat{v}(0)$ and $\mathcal{H}$ the (suitably normalized) Hilbert transform

$$\mathcal{H}v(s) = -\frac{1}{2\pi} p.v. \int_0^1 \cot(\pi(s-\sigma))v(\sigma)\, d\sigma.$$

Therefore, the operator M of (2.6) takes the form

$$M = -\mathcal{H}DB + \mathcal{J}B. \tag{2.8}$$

From [4, 5] we now recall some analytical results on Equations (1.5) and (2.6) which are needed in the convergence analysis of the trigonometric collocation method. The first theorem was proved in [4], using a decomposition of M into a Mellin convolution operator local to the corner and a compact operator on H^0.

Theorem 2.1 *The operators $I + M : H^0 \to H^0$ and $K : H^0 \to H^1$ are continuously invertible, and we have the strong ellipticity estimate*

$$Re((I + M + T)v, v) \geq c\|v\|_0^2, \quad v \in H^0,$$

with some compact operator T on H^0.

The next result, also taken from [4], shows that the unique solution of (1.5) is smooth provided the right-hand side of (1.1) is smooth and the grading exponent is sufficiently large. Let $H^l(\Gamma), l > 0$, denote the restriction of the usual Sobolev space $H^{l+1/2}(\mathbb{R}^2)$ to Γ.

Theorem 2.2 *Let $l \in \mathbb{N}, q > (l+1/2)(1+|\chi|)$, and suppose that $f \in H^{l+5/2}(\Gamma)$. Then the unique solution of (1.5) satisfies $w \in H^l$. Moreover, there exists $\delta < 1/2$ such that*

$$D^m w(s) = O(|s|^{l-m-\delta}) \quad \text{as } s \to 0, \quad m = 0, ..., l. \tag{2.9}$$

The following result from [5] describes the properties of the kernel function $b(s, \sigma)$ defined in (1.10).

Theorem 2.3 *On each compact subset of $\mathbb{R} \times \mathbb{R} \setminus (\mathbb{Z} \times \mathbb{Z})$, the derivatives $D_s^i D_\sigma^m b(s, \sigma)$ of order $i + m \leq q$ are bounded and 1-periodic. Moreover, for $s, \sigma \in [-1/2, 1/2] \setminus \{0\}$, we have the estimates*

$$|b(s,\sigma)| \leq c|\log(|s| + |\sigma|)|,$$

$$|D_s^i D_\sigma^m b(s,\sigma)| \leq c(|s| + |\sigma|)^{-i-m}, \quad 1 \leq i + m \leq q.$$

Let us now consider the collocation method (2.5). We shall rewrite this as a projection method for (2.6). For any $v \in H^0$, let $R_h v \in \mathcal{T}_h$ solve the collocation equation $AR_h v = Q_h Av$. Then $R_h = A^{-1} Q_h A$ is a well defined projection operator of H^0 onto $\mathcal{T}_h$ which satisfies (see (2.4))

$$\|v - R_h v\|_t \le ch^{r-t} \|v\|_r, \quad v \in H^r, \quad \text{for } r > -1/2, \quad r \ge t \ge -1. \tag{2.10}$$

It is then straightforward to see that (2.5) is equivalent to

$$(I + R_h M) w_h = R_h e.$$

As is usual for Mellin convolution operators, we are only able to prove stability for a slightly modified method. Introduce, for $\tau > 0$ sufficiently small, the truncation $T_\tau v$ as the 1-periodic extension of

$$T_\tau v(s) = \begin{cases} v(s), & s \in (\tau, 1-\tau), \\ 0, & s \in (0,\tau) \cup (1-\tau, 1) \end{cases}$$

and consider the modified collocation method

$$(A + Q_h B T_{i^* h}) w_h = Q_h g, \qquad w_h \in \mathcal{T}_h, \tag{2.11}$$

where i^* is a fixed natural number independent of h. If $i^* = 0$ then (2.11) coincides with (2.5). Otherwise, (2.11) can be obtained from (2.5) by a slight change to the coefficient matrix of the corresponding linear system. Now it is easily seen that (2.11) is equivalent to

$$(I + R_h M T_{i^* h}) w_h = R_h e, \qquad w_h \in \mathcal{T}_h. \tag{2.12}$$

The following theorem establishes the convergence of the (modified) collocation method with optimal order in the L^2 norm.

Theorem 2.4 *Let $q \ge 2$, and suppose that i^* is sufficiently large.*
(i) The method (2.12) is stable, that is the estimate

$$\|(I + R_h M T_{i^* h}) v\|_0 \ \ge c \|v\|_0, \qquad v \in \mathcal{T}_h \tag{2.13}$$

holds for all h sufficiently small, where c is independent of h and v.
(ii) If, in addition, the hypothesis of Theorem 2.2 holds, then (2.11) has a unique solution for all h sufficiently small and

$$\|w - w_h\|_0 \le ch^l, \tag{2.14}$$

where c is a constant which depends on w and i^* but is independent of h.

Proof: Following [4, Theorem 9], we first verify the stability estimate (2.13). Since, by Theorem 2.1, $I + M$ is strongly elliptic and invertible on H^0, we obtain stability of the finite section operators $T_\tau(I + M)T_\tau$ as $\tau \to 0$, which implies the estimate (see [4, Theorem 6])

$$\|(I + MT_\tau)v\|_0 \geq c\|v\|_0, \qquad v \in H^0, \quad \tau \leq \tau_0. \tag{2.15}$$

Then (2.13) is obtained with the help of (2.15) and the following perturbation result.

For fixed $q \geq 2$ and each $\epsilon > 0$, there exists $i^* \geq 1$ such that for all h sufficiently small

$$\|(I - R_h)MT_{i^*h}v\|_0 \leq \epsilon\|v\|_0, \qquad v \in H^0. \tag{2.16}$$

From (2.8) and (2.10) and the fact that $I - R_h$ annihilates the constants, we obtain the estimate

$$\|(I - R_h)MT_{i^*h}v\|_0 \leq ch\|DMT_{i^*h}v\|_0 \leq ch\|D^2BT_{i^*h}v\|_0, \qquad v \in H^0.$$

To prove (2.16), it is now sufficient to verify that

$$\|D^2BT_{i^*h}v\|_0 \leq (c/i^*h)\|v\|_0, \qquad v \in H^0. \tag{2.17}$$

where c is independent of i^*, h and v. Using Theorem 2.3, we now obtain

$$\begin{aligned}
|D^2BT_{i^*h}v(s)| &\leq \int_{J_{i^*h}} |D_s^2 b(s,\sigma)||v(\sigma)|\, d\sigma \\
&\leq c\int_{J_{i^*h}} (|s| + |\sigma|)^{-2}\ |v(\sigma)|\, d\sigma \\
&\leq (c/i^*h)\int_{J_{i^*h}} \frac{|\sigma|}{(|s| + |\sigma|)^2}\ |v(\sigma)|\, d\sigma, \qquad s \in (-1/2, 1/2),
\end{aligned}$$

where $J_{i^*h} = (-1/2, -i^*h) \cup (i^*h, 1/2)$. Taking L^2 norms and using the fact that the integral operator with Mellin convolution kernel $(s+\sigma)^{-1}$ is bounded on $L^2(0,\infty)$ then gives (2.17).

To prove the error estimate (2.14), we note that

$$\|w - w_h\|_0 \leq \|(I - R_h)w\|_0 + \|w_h - R_h w\|_0,$$

where the first term is of order h^l by Theorem 2.2 and (2.10) (with $t = 0, r = l$). Furthermore, using (2.13) and then (2.12) with (2.6) and the uniform boundedness of R_h on H^0, we obtain

$$\|w_h - R_h w\|_0 \quad \leq \quad c\|(I + R_h MT_{i^*h})(w_h - R_h w)\|_0$$

$$
\begin{aligned}
&= c\|R_h[(I+M)w-(I+MT_{i^*h})R_hw]\|_0 \\
&\leq c\|(I+MT_{i^*h})(I-R_h)w+M(I-T_{i^*h})w\|_0 \\
&\leq c\|(I-R_h)w\|_0+c\|(I-T_{i^*h})w\|_0.
\end{aligned}
$$

The proof is complete since by (2.9) (with $m=0$) the last term is of order h^l again. □

The following corollary shows that the collocation solutions to the transformed equation yield superconvergent approximations to interior potentials.

Corollary 2.5 *Under the hypothesis of Theorem 2.4 (ii), we have*

$$\|w-w_h\|_{-1} \leq ch^{l+\beta}$$

where $\beta=1$ *if* $i^*=0$ *and* $\beta=1/2$ *if* $i^* \geq 1$.

Proof: We restrict ourselves to the case of the unmodified method; for $i^* \geq 1$ we refer to [7]. Suppose that (2.13) holds with $i^*=0$.

Let $v \in H^1$, and write $v=Kv_1$ with $v_1 \in H^0$. Then, since $Q_hKw_h=Q_hKw$,

$$(w-w_h,v)=(w-w_h,Kv_1)=(K(w-w_h),v_1)$$

$$=((I-Q_h)K(w-w_h),v_1).$$

Hence, by (2.4), Theorem 2.1 and (2.14)

$$|(w-w_h,v)| \leq ch\|K(w-w_h)\|_1\|v_1\|_0 \leq ch\|w-w_h\|_0\|v\|_1 \leq ch^{l+1}\|v\|_1,$$

which proves the result. □

3 Discrete Collocation

To define a fully discrete version of the collocation method (2.5), introduce the nodes

$$\sigma_j = jh, \qquad j \in \mathbb{Z}, \quad \text{where} \quad h:=1/(2n+1). \tag{3.1}$$

To evaluate the integral

$$I(v)=\int_0^1 v(\sigma)\,d\sigma$$

for a 1-periodic continuous function v, approximate it by the trapezoidal rule

$$I_h(v)=h\sum_{j=0}^{2n} v(\sigma_j). \tag{3.2}$$

The integral operator B of (1.9) is now approximated by

$$B_h v(s) := I_h(b(s,\cdot)v(\cdot)) = h \sum_{j=0}^{2n} b(s,\sigma_j)v(\sigma_j), \tag{3.3}$$

and replacing B with B_h in (2.5), the discrete collocation method can be written in the form

$$(A + Q_h B_h)w_h = Q_h g, \qquad w_h \in \mathcal{T}_h. \tag{3.4}$$

To obtain a linear system for finding w_h, let

$$w_h(s) = \sum_{k=-n}^{n} \alpha_k \varphi_k(s)$$

and calculate the coefficients α_k from (3.4) and the definitions of A, Q_h and B_h:

$$\sum_{k=-n}^{n} \alpha_k \left[\frac{\varphi_k(s_j)}{\max(1,|k|)} + (B_h \varphi_k)(s_j) \right] = g(s_j), \qquad j = 0, \dots, 2n. \tag{3.5}$$

Our convergence analysis follows the same lines as in § 2. That is, instead of (3.4) we consider the modified method

$$(A + Q_h B_h T_{i^*h})w_h = Q_h g, \qquad w_h \in \mathcal{T}_h. \tag{3.6}$$

Setting $M_h = A^{-1}B_h$ and using (2.6) and the projection R_h defined in § 2, (3.6) can be written as

$$(I + R_h M_h T_{i^*h})w_h = R_h e, \qquad w_h \in \mathcal{T}_h. \tag{3.7}$$

For our analysis, the following standard estimate for the trapezoidal rule (3.2) is needed.

Lemma 3.1 *Let $l \in \mathbb{N}$, and suppose that v has 1-periodic continuous derivatives of order $< l$ on $\mathbb{R}$ and that $D^l v$ is integrable on $(0,1)$. Then*

$$|I(v) - I_h(v)| \le ch^l \int_0^1 |D^l v(\sigma)| d\sigma,$$

where c does not depend on v and h.

The proof of Lemma 3.1 is based on the representation

$$I(v) - I_h(v) = h^l \int_0^1 P_l(\sigma/h) D^l v(\sigma)\, d\sigma,$$

where P_l is some 1-periodic piecewise polynomial of degree l, see [3, Chap. 2.9].

The following lemma is the key to the stability of (3.7).

Lemma 3.2 *For fixed $q \geq 2$ and for each $\epsilon > 0$, there exists $i^* \geq 1$ independent of h such that, for all $v \in \mathcal{T}_h$ and all sufficiently small h,*

$$\|(M - M_h)T_{i^*h}M_hT_{i^*h}v\|_0 \leq \epsilon\|v\|_0, \tag{3.8}$$

$$\|(M - M_h)T_{i^*h}(I - R_h)M_hT_{i^*h}v\|_0 \leq \epsilon\|v\|_0. \tag{3.9}$$

Proof: We first show the estimate

$$\|(M - M_h)T_{i^*h}u\|_0 \leq (c/i^*)\|u\|_0 + ch\|Du\|_0, \qquad u \in H^1, \tag{3.10}$$

where c is independent of i^*, h and u. Using (2.7) and the definition of M_h, we have

$$\|(M - M_h)T_{i^*h}u\|_0 \leq c\{\|(B - B_h)T_{i^*h}u\|_0 + \|D(B - B_h)T_{i^*h}u\|_0\}. \tag{3.11}$$

Furthermore, using Lemma 3.1 (for $l = 1$ and the interval $(-1/2, 1/2)$) and Theorem 2.3, we obtain

$$\begin{aligned}
& |(B - B_h)T_{i^*h}u(s)| + |D(B - B_h)T_{i^*h}u(s)| \\
\leq\ & ch\int_{J_{i^*h}} \{|b(s,\sigma)||u'(\sigma)| + |D_\sigma b(s,\sigma)||u(\sigma)|\}\, d\sigma \\
+\ & ch\int_{J_{i^*h}} \{|D_s b(s,\sigma)||u'(\sigma)| + |D_s D_\sigma b(s,\sigma)||u(\sigma)|\}\, d\sigma \\
\leq\ & ch\int_{J_{i^*h}} \frac{1}{(|s| + |\sigma|)^2}|u(\sigma)|d\sigma + ch\int_{J_{i^*h}} \frac{|u'(\sigma)|}{|s| + |\sigma|}d\sigma
\end{aligned}$$

$$\leq (c/i^*)\int_{J_{i^*h}} \frac{|\sigma|}{(|s| + |\sigma|)^2}|u(\sigma)|d\sigma + ch\int_{J_{i^*h}} \frac{|u'(\sigma)|}{|s| + |\sigma|}d\sigma, \quad s \in (-1/2, 1/2), \tag{3.12}$$

where $J_{i^*h} = (-1/2, -i^*h) \cup (i^*h, 1/2)$. Taking L^2 norms in (3.12) and using the fact that an integral operator with Mellin convolution kernel $\sigma^m/(s + \sigma)^{m+1}, m \geq 0$, is bounded on $L^2(0, \infty)$ gives (3.10).

To complete the proof of (3.8), we set $u = M_hT_{i^*h}v$ in (3.10). Then we have to verify that, for all $v \in \mathcal{T}_h$,

$$\|M_hT_{i^*h}v\|_0 \leq c\|v\|_0, \tag{3.13}$$

$$\|DM_hT_{i^*h}v\|_0 \leq (c/i^*h)\|v\|_0. \tag{3.14}$$

Since MT_{i^*h} is obviously uniformly bounded on H^0, it suffices to prove (3.13) with M_h replaced by $M - M_h$. Applying estimate (3.10) again, we then get

$$\|(M - M_h)T_{i^*h}v\|_0 \leq (c/i^*)\|v\|_0 + ch\|Dv\|_0 \leq c\|v\|_0, \qquad v \in \mathcal{T}_h, \tag{3.15}$$

where we have used the inverse property of $\mathcal{T}_h$ (Bernstein's inequality); see e.g. [10, Chap. 2.1]. To prove (3.14), we observe that

$$\|DM_hT_{i^*h}v\|_0 \le \|DMT_{i^*h}v\|_0 + \|D(M-M_h)T_{i^*h}v\|_0. \tag{3.16}$$

As in the proof of (2.16), the first term of (3.16) can be bounded by $(c/i^*h)\|v\|_0$. Analogously to (3.11) and (3.12), we have from (2.7), Lemma 3.1 and Theorem 2.3

$$\|D(M-M_h)T_{i^*h}v\|_0 \le c\|D^2(B-B_h)T_{i^*h}v\|_0$$

and

$$\begin{aligned}
&|D^2(B-B_h)T_{i^*h}v(s)| \\
&\le ch\int_{J_{i^*h}} \{|D_s^2D_\sigma b(s,\sigma)||v(\sigma)| + |D_s^2 b(s,\sigma)||v'(\sigma)|\}\, d\sigma \\
&\le ch\int_{J_{i^*h}} \frac{1}{(|s|+|\sigma|)^3}|v(\sigma)|\, d\sigma + ch\int_{J_{i^*h}} \frac{1}{(|s|+|\sigma|)^2}|v'(\sigma)|\, d\sigma \\
&\le ch(i^*h)^{-2}\int_{J_{i^*h}} \frac{\sigma^2}{(|s|+|\sigma|)^3}|v(\sigma)|\, d\sigma + ch(i^*h)^{-1}\int_{J_{i^*h}} \frac{|\sigma|}{(|s|+|\sigma|)^2}|v'(\sigma)|\, d\sigma
\end{aligned}$$

for any $s \in (-1/2, 1/2)$. Taking L^2 norms and applying the inverse property of $\mathcal{T}_h$ then gives

$$\|D(M-M_h)T_{i^*h}v\|_0 \le (c/i^*h)\|v\|_0 + (c/i^*)\|v'\|_0 \le (c/i^*h)\|v\|_0,$$

which completes the proof of (3.14). Therefore, (3.8) follows from (3.10), (3.13) and (3.14) provided i^* is chosen large enough.

To prove (3.9), we now set $u = (I-R_h)M_hT_{i^*h}v$ in (3.10) to obtain the estimate

$$\begin{aligned}
&\|(M-M_h)T_{i^*h}(I-R_h)M_hT_{i^*h}v\|_0 \\
&\le (c/i^*)\|(I-R_h)M_hT_{i^*h}v\|_0 + ch\|D(I-R_h)M_hT_{i^*h}v\|_0.
\end{aligned}$$

Using (2.10), together with the fact that $I-R_h$ annihilates the constants, and (3.14), the last expression can further be bounded by

$$\begin{aligned}
&(ch/i^*)\|DM_hT_{i^*h}v\|_0 + ch\|DM_hT_{i^*h}v\|_0 \\
&\le ch\|DM_hT_{i^*h}v\|_0 \le (c/i^*)\|v\|_0,
\end{aligned}$$

which gives the result by choosing i^* sufficiently large. □

We are now in the position to prove stability for the method (3.7).

Theorem 3.3 *Assume $q \geq 2$, and suppose that i^* is sufficiently large. Then the estimate*

$$\|(I + R_h M_h T_{i^*h})v\|_0 \geq c\|v\|_0, \qquad v \in \mathcal{T}_h \tag{3.17}$$

holds for all h sufficiently small, where c is independent of v and h.

Proof: By Theorem 2.4 (i), the operators

$$(I + R_h M T_{i^*h})^{-1} : \mathcal{T}_h \to \mathcal{T}_h, \qquad h \leq h_0$$

exist and are uniformly bounded with respect to the H^0 operator norm if i^* is large enough. Consider the operators

$$C_h := I - (I + R_h M T_{i^*h})^{-1} R_h M_h T_{i^*h}$$

and

$$D_h := (I + R_h M T_{i^*h})^{-1} R_h (M_h - M) T_{i^*h} R_h M_h T_{i^*h}$$

acting on $\mathcal{T}_h$. Then simple computation shows that

$$C_h(I + R_h M_h T_{i^*h}) = I - D_h. \tag{3.18}$$

Using (3.15) and the uniform boundedness of R_h on H^0, we observe that $R_h M_h T_{i^*h}$ and hence C_h are uniformly bounded, too. Furthermore, from Lemma 3.2 we obtain for some $\epsilon \in (0,1)$

$$\|D_h v\|_0 \leq c\|R_h(M_h - M)T_{i^*h} R_h M_h T_{i^*h} v\|_0$$

$$\leq c\{\|(M - M_h)T_{i^*h} M_h T_{i^*h} v\|_0 + \|(M - M_h)T_{i^*h}(I - R_h)M_h T_{i^*h} v\|_0\}$$

$$\leq \epsilon\|v\|_0, \qquad v \in \mathcal{T}_h, \quad h \leq h_0$$

provided that i^* is sufficiently large. Therefore, $(I - D_h)$ is invertible with $(I - D_h)^{-1}$ uniformly bounded, and (3.18) implies the estimate

$$\|(I + R_h M T_{i^*h})^{-1}\|_0 \leq \|(I - D_h)^{-1}\|_0 \; \|C_h\|_0 \leq c, \qquad h \leq h_0,$$

which proves (3.17). □

Finally, we show that (3.6) converges with the same optimal order as the collocation method considered in § 2.

Theorem 3.4 *Let $l \in I\!N, q > (l+1/2)(1+|\chi|)$, and suppose that $f \in H^{l+5/2}(\Gamma)$. Suppose further that i^* is sufficiently large. Then (3.6) has a unique solution for all h sufficiently small and*

$$\|w - w_h\|_0 \leq ch^l, \tag{3.19}$$

where c is independent of h.

Proof: As in the proof of Theorem 2.4 (ii), we have

$$\|w - w_h\| \leq \|(I - R_h)w\|_0 + \|w_h - R_h w\|_0,$$

where the first term is of order h^l. Using Theorem 3.3 and the uniform boundedness of R_h, we get

$$\begin{aligned} \|w_h - R_h w\|_0 &\leq c\|(I + R_h M_h T_{i^*h})(w_h - R_h w)\|_0 \\ &= c\|R_h(I+M)w - R_h(I + M_h T_{i^*h})R_h w)\|_0 \\ &\leq c\|Mw - M_h T_{i^*h} R_h w\|_0 \end{aligned}$$

$$\leq c\|(M - M_h T_{i^*h})w\|_0 + c\|M_h T_{i^*h}(w - R_h w)\|_0. \tag{3.20}$$

To estimate the second term on the right side of (3.20), we observe that

$$\|M_h T_{i^*h}(w - R_h w)\|_0 \leq \|M T_{i^*h}(I - R_h)w\|_0 + \|(M - M_h)T_{i^*h}(I - R_h)w\|_0$$

$$\leq c\|(I - R_h)w\|_0 + ch\|D(I - R_h)w\|_0,$$

where we have used (3.10). Since $w \in H^l$ (see Theorem 2.2), the last two terms are of order h^l by virtue of (2.10).

To complete the estimate (3.19), we need an analogous bound for the first term in (3.20). Note that

$$\|(M - M_h T_{i^*h})w\|_0 \leq \|M(I - T_{i^*h}w\|_0 + \|(M - M_h)T_{i^*h})w\|_0$$

$$\leq c\|(I - T_{i^*h})w\|_0 + \|(M - M_h)T_{i^*h}w\|_0. \tag{3.21}$$

Since Theorem 2.2 implies $w(s) = O(|s|^{l-1/2})$ as $s \to 0$, the first term is of order h^l. To estimate the last term of (3.21), we proceed as in (3.12) and apply Lemma 3.1 and Theorem 2.3 to obtain

$$|(B - B_h)T_{i^*h}w(s)| + |D(B - B_h)T_{i^*h}w(s)|$$

$$
\begin{aligned}
&\leq \; ch^l \int_{J_{i^*h}} \sum_{m=0}^{l} \left(|D_\sigma^m b(s,\sigma)| + |D_s D_\sigma^m b(s,\sigma)|\right) |D^{l-m} w(\sigma)| \, d\sigma \\
&\leq \; ch^l \int_{J_{i^*h}} \sum_{m=0}^{l} \frac{1}{(|s|+|\sigma|)^{m+1}} |D^{l-m} w(\sigma)| \, d\sigma \\
&\leq \; ch^l \sum_{l=0}^{l} \int_{J_{i^*h}} \frac{|\sigma|^m}{(|s|+|\sigma|)^{m+1}} |\sigma|^{-m} |D^{l-m} w(\sigma)| d\sigma
\end{aligned}
$$

for any $s \in (-1/2, 1/2)$. Taking L^2 norms and using (3.11) then gives

$$
\|(M - M_h) T_{i^*h} w\|_0 \leq ch^l \sum_{m=0}^{l} \; \||s|^{-m} D^{l-m} w\|_0 \leq ch^l,
$$

since the estimate (2.9) ensures that

$$
|s|^{-m} D^{l-m} w \in H^0, \qquad m = 0, ..., l.
$$

This finishes the proof of (3.19). □

4 Numerical results

In this section we consider a numerical example illustrating the solution of Equation (1.1) when Γ is given by

$$
\gamma_0(s) = \sin \pi s (\cos(1-\chi)\pi s, \sin(1-\chi)\pi s)^T, \quad s \in [0,1], \quad \chi \in (0,1).
$$

In this example (also described in [4, 5]), Γ is the boundary of a "teardrop-shaped" region with a single corner at $s = 0$ (or $s = 1$) and smooth elsewhere. The exterior angle between the tangents at $s = 0$ and $s = 1$ is $(1+\chi)\pi$. Since it will be a straightforward technical matter to extend the results of this paper to the case of a curvilinear polygon, this example provides a reasonable test of our theoretical results. With this parametrization we put (1.1) in the form (1.5) and solved the latter equation numerically using the quadrature-collocation scheme (3.4). We took the right-hand side f to be

$$
f(\mathbf{x}) = \exp(x_1)\cos(x_2) + \mathrm{Re}\{(x_1 + ix_2)^{1/(1-\chi)}\}.
$$

Then f is the Dirichlet data for a harmonic function in the interior of Γ which has the (weak) singularity induced by the corner in Γ. Nevertheless the solution of (1.1) will have the stronger singularity to the worst of the two singularities appearing in the exterior and interior harmonic boundary value problem for this domain. Theorem 3.4 implies that the numerical solution w_h will converge to the true solution with rate

$$\|w - w_h\|_0 = O(h^k), \tag{4.1}$$

provided

$$q > (k + 1/2)(1 + \chi). \tag{4.2}$$

In all experiments of this section, no modification of the collocation method was found necessary for stability and throughout we have set $i^* = 0$. All experiments shown here are for $\chi = 0.76$.

Special care must be taken in the implementation of (3.5), in particular in the evaluation of

$$b(s_j, \sigma_i) = -2 \log \left| \frac{\gamma(s_j) - \gamma(\sigma_i)}{2e^{-1/2} \sin \pi(s_j - \sigma_i)} \right| \tag{4.3}$$

when the numerator and the denominator are both close to zero. For large n, large grading parameter q and in the extreme case when s_j and σ_i are near the endpoints of $[0, 1]$, the numerator on the right hand side of (4.3) becomes much closer to zero than the denominator and numerically (4.3) becomes $\log(0)$. In this case an appropriate limiting approximation had to be used. For further details see [5].

The exact solution w of (1.5) is unknown. To check (4.1) we computed an approximation w^* using $n = 512$ and used that as the exact solution. Then $\|w^* - w_h\|_0$ was computed exactly using Parseval's equality. Empirically determined convergence rates are given in columns headed "EOC" in Table 1 and demonstrate the expected improvement of the convergence order for increasing values of the grading exponent q. Note that from (4.1) and (4.2) one would predict the convergence rates 0.64, 1.20, 1.77, 2.34 corresponding to $q = 2, 3, 4, 5$, respectively.

Table 1

	$q=2$		$q=3$		$q=4$		$q=5$	
n	$\lVert w_h - w^* \rVert_0$	EOC	$\lVert w_h - w^* \rVert_0$	EOC	$\lVert w_h - w^* \rVert_0$	EOC	$\lVert w_h - w^* \rVert_0$	EOC
8	5.91 -3		4.61 -3		2.27 -3		5.28 -3	
		0.76		1.17		2.33		5.26
16	3.48 -3		2.05 -3		4.51 -4		1.38 -4	
		0.68		1.18		1.73		2.41
32	2.17 -3		9.01 -4		1.35 -4		2.58 -5	
		0.73		1.20		1.75		2.35
64	1.31 -3		3.90 -4		4.01 -5		5.03 -6	
		0.86		1.25		1.74		2.32
128	7.21 -4		1.63 -4		1.20 -5		1.01 -6	
		1.26		1.38		1.72		2.38
256	3.01 -4		6.26 -5		3.63 -6		1.93 -7	

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